

TABLEAUX COMBINATORICS FOR THE ASYMMETRIC EXCLUSION PROCESS

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ABSTRACT. The partially asymmetric exclusion process (PASEP) is an important model from statistical mechanics which describes a system of interacting particles hopping left and right on a one-dimensional lattice of n sites. It is partially asymmetric in the sense that the probability of hopping left is q times the probability of hopping right. Additionally, particles may enter from the left with probability α and exit from the right with probability β .

In this paper we prove a close connection between the PASEP and the combinatorics of permutation tableaux. (These tableaux come indirectly from the totally nonnegative part of the Grassmannian, via work of Postnikov, and were studied in a paper of Steingrímsson and the second author.) Namely, we prove that in the long time limit, the probability that the PASEP is in a particular configuration τ is essentially the generating function for permutation tableaux of shape $\lambda(\tau)$ enumerated according to three statistics. The proof of this result uses a result of Derrida, Evans, Hakim, and Pasquier on the *matrix ansatz* for the PASEP model.

As an application, we prove some monotonicity results for the PASEP. We also derive some enumerative consequences for permutations enumerated according to various statistics such as weak excedence set, descent set, crossings, and occurrences of generalized patterns.

1. INTRODUCTION

The partially asymmetric exclusion process (PASEP) is an important model from statistical mechanics which is quite simple but surprisingly rich: it exhibits boundary-induced phase transitions, spontaneous symmetry breaking, and phase separation. The PASEP is regarded as a primitive model for biopolymerization [11], traffic flow [15], and formation of shocks [8]; it also appears in a kind of sequence alignment problem in computation biology [4]. More recently it has been noticed that the PASEP model has relations to orthogonal polynomials [13], and to interesting combinatorial phenomena [9, 5, 14]. The goal of this paper is to prove a precise connection between the PASEP model and permutation tableaux, certain $0-1$ tableaux introduced in [16].

The PASEP model describes a system of particles hopping left and right on a one-dimensional lattice of n sites. Particles may enter the system from the left with

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a rate αdt and may exit the system from the right at a rate βdt . The probability of hopping left is q times the probability of hopping right.

Let $\frac{f_\tau(q)}{Z_n}$ denote the probability that the PASEP model is in a particular configuration τ in the steady state. Here, Z_n is the *partition function* of the PASEP model. In this paper we consider a solution (D_1, E_1, V_1, W_1) to the “matrix ansatz” for the PASEP model which naturally relates to permutation tableaux. That is, we find matrices D_1 and E_1 and vectors V_1 and W_1 which satisfy the relations of Theorem 2.3; we then prove that expressions of the form $W_1(\prod_{i=1}^n (\tau_i D_1 + (1 - \tau_i) E_1)) V_1$, where $\tau = (\tau_1, \dots, \tau_n) \in \{0, 1\}^n$, are (Laurent) polynomials in q, α, β which enumerate permutation tableaux of a fixed shape $\lambda(\tau)$ according to three statistics. Using a result of Derrida et al [7], we are then able to prove our main result: that $f_\tau(q)$ is the generating function for permutation tableaux of a fixed shape $\lambda(\tau)$.

It then follows from work of the second author and Steingrímsson [16] that $f_\tau(q)$ is also the generating function for: permutations in S_{n+1} with a fixed set $W(\tau)$ of weak excedences, enumerated according to crossings; permutations in S_{n+1} with a fixed set $D(\tau)$ of descents, enumerated according to occurrences of the *generalized pattern 2–31*. Additionally, these results imply that the expression $W_1(D_1 + E_1)^n V_1$ for the partition function Z_n is also the weight-generating function for permutations in S_{n+1} , enumerated according to crossings, and permutations in S_{n+1} , enumerated according to occurrences of the *generalized pattern 2–31*.

Our main result refines the theorem of the first author [5], who showed that if $\alpha = \beta = 1$, then in the steady state, the probability that the model is in a configuration with k occupied sites is equal to $\frac{\hat{E}_{k+1, n+1}(q)}{Z_n}$, where $\hat{E}_{k+1, n+1}(q)$ is the q -Eulerian polynomial introduced by the second author [17].

The structure of this paper is as follows. In Section 2 we define the PASEP model and review the “matrix ansatz,” presenting a solution (D_1, E_1) which has an interpretation in terms of *permutation tableaux*. In Section 3 we define permutation tableaux, certain 0–1 tableaux which were defined in [16] and which are naturally in bijection with permutations. We then show how the solution (D_1, E_1) to the matrix ansatz leads to a natural connection between the PASEP model and permutation tableaux, and hence to permutations. In Section 4, we prove the main result of the previous section, and in Section 5, we give some applications. Finally, in section 6, we show how a classical solution (D_0, E_0) of the matrix ansatz leads to a natural connection between the PASEP model and bicolored Motzkin paths. This recovers results of Brak et al [3].

It is interesting to note that permutation tableaux are closely connected to total positivity for the Grassmannian [17, 12]. This suggests an intriguing connection between total positivity and the PASEP model.

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2. THE PASEP MODEL AND THE “MATRIX ANSATZ”

In the physics literature, the PASEP is defined as follows.

Definition 2.1. *We are given a one-dimensional lattice of N sites, such that each site i ($1 \leq i \leq N$) is either occupied by a particle ($\tau_i = 1$) or is empty ($\tau_i = 0$). At most one particle may occupy a given site. During each infinitesimal time interval dt , each particle in the system has a probability dt of jumping to the next site on its right (for particles on sites $1 \leq i \leq N-1$) and a probability qdt of jumping to the next site on its left (for particles on sites $2 \leq i \leq N$). Furthermore, a particle is added at site $i = 1$ with probability αdt if site 1 is empty and a particle is removed from site N with probability βdt if this site is occupied.*

Note that we will sometimes denote a state of the PASEP as a word in $\{0, 1\}^N$ and sometimes as a word in $\{\circ, \bullet\}^N$. In the latter notation, the symbol \circ denotes the absence of a particle, which one can also think of as a white particle.

It is not too hard to see [9] that our previous formulation of the PASEP is equivalent to the following discrete-time Markov chain.

Definition 2.2. *Let α , β , and q be constants such that $0 < \alpha \leq 1$, $0 < \beta \leq 1$, and $0 \leq q \leq 1$. Let B_N be the set of all 2^N words in the language $\{\circ, \bullet\}^*$. The PASEP is the Markov chain on B_N with transition probabilities:*

- If $X = A \bullet \circ B$ and $Y = A \circ \bullet B$ then $P_{X,Y} = \frac{1}{N+1}$ (particle hops right) and $P_{Y,X} = \frac{q}{N+1}$ (particle hops left).
- If $X = \circ B$ and $Y = \bullet B$ then $P_{X,Y} = \frac{\alpha}{N+1}$ (particle enters from left).
- If $X = B \bullet$ and $Y = B \circ$ then $P_{X,Y} = \frac{\beta}{N+1}$ (particle exits to the right).
- Otherwise $P_{X,Y} = 0$ for $Y \neq X$ and $P_{X,X} = 1 - \sum_{Y \neq X} P_{X,Y}$.

See Figure 1 for an illustration of the four states, with transition probabilities, for the case $n = 2$. Note that the probabilities of the loops are determined from the figure by the fact that the sum of the probabilities on all outgoing arrows from a given state must be 1. So for example the probability on the bottom-most loop is $1 - \frac{q}{3} - \frac{\alpha}{3} - \frac{\beta}{3} = \frac{3-q-\alpha-\beta}{3}$.

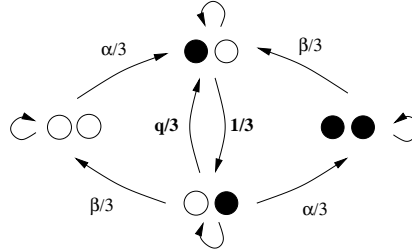


FIGURE 1. The state diagram of the PASEP model for $n = 2$

In the long time limit, the system reaches a steady state where all the probabilities $P_n(\tau_1, \tau_2, \dots, \tau_n)$ of finding the system in configurations $(\tau_1, \tau_2, \dots, \tau_n)$ are

stationary, i.e. satisfy

$$\frac{d}{dt}P_n(\tau_1, \dots, \tau_n) = 0.$$

Moreover, the stationary distribution is unique [7], as shown by Derrida et al.

The question is now to solve for the probabilities $P_n(\tau_1, \dots, \tau_n)$. For convenience, we define unnormalized weights $f_n(\tau_1, \dots, \tau_n)$, which are equal to the $P_n(\tau_1, \dots, \tau_n)$ up to a constant:

$$P_n(\tau_1, \dots, \tau_n) = f_n(\tau_1, \dots, \tau_n)/Z_n,$$

where Z_n is the *partition function* $\sum_{\tau} f_n(\tau_1, \dots, \tau_n)$. The sum defining Z_n is over all possible configurations $\tau \in \{0, 1\}^n$.

The “matrix ansatz” has been used by Derrida et al [7] to obtain exact expressions for all the $P_n(\tau_1, \dots, \tau_n)$.

More precisely, they show the following.

Theorem 2.3. [7] *Suppose that D and E are matrices, V is a column vector, and W is a row vector, such that the following conditions hold:*

$$\begin{aligned} DE - qED &= D + E \\ DV &= \frac{1}{\beta}V \\ WE &= \frac{1}{\alpha}W \end{aligned}$$

Then

$$f_n(\tau_1, \dots, \tau_n) = W\left(\prod_{i=1}^n (\tau_i D + (1 - \tau_i)E)\right)V.$$

Note that $\prod_{i=1}^n (\tau_i D + (1 - \tau_i)E)$ is simply a product of n matrices D or E with matrix D at position i if site i is empty ($\tau_i = 0$).

Remark 2.4. *It follows from Theorem 2.3 that the partition function Z_n is equal to $W(D + E)^n V$.*

We will now describe a solution (D_1, E_1, V_1, W_1) to the matrix ansatz. It seems to have not been considered before, although it naturally generalizes a solution given in [7] for the $q = 0$ case. Our solution has an interpretation in terms of permutation tableaux and we found it through consideration of the combinatorics of these tableaux; however, one could also arrive at these matrices by choosing the basis $\{V_1, EV_1, E^2V_1, \dots\}$ for the infinite-dimensional vector space that D and E act on, and then writing D and E according to this basis. (This method actually yields the transposes of the matrices D_1 and E_1 which we use below.)

Let D_1 be the (infinite) upper triangular matrix (d_{ij}) such that $d_{i,i+1} = \beta^{-1}$ and $d_{i,j} = 0$ for $j \neq i + 1$.

That is, D_1 is the matrix

$$\begin{pmatrix} 0 & \beta^{-1} & 0 & 0 & \dots \\ 0 & 0 & \beta^{-1} & 0 & \dots \\ 0 & 0 & 0 & \beta^{-1} & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let E_1 be the (infinite) lower triangular matrix (e_{ij}) such that for $j \leq i$, $e_{ij} = \beta^{i-j}(\alpha^{-1}q^{j-1}\binom{i-1}{j-1} + \sum_{r=0}^{j-2}\binom{i-j+r}{r}q^r)$. Otherwise, $e_{ij} = 0$.

That is, E_1 is the matrix

$$\begin{pmatrix} \alpha^{-1} & 0 & 0 & 0 & \dots \\ \alpha^{-1}\beta & 1 + \alpha^{-1}q & 0 & 0 & \dots \\ \alpha^{-1}\beta^2 & \beta(1 + 2\alpha^{-1}q) & 1 + q + \alpha^{-1}q^2 & 0 & \dots \\ \alpha^{-1}\beta^3 & \beta^2(1 + 3\alpha^{-1}q) & \beta(1 + 2q + 3\alpha^{-1}q^2) & 1 + q + q^2 + \alpha^{-1}q^3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Observe that when $\alpha = \beta = 1$, we have $e_{ij} = \frac{[i]^{(i-j)}}{(i-j)!}$. Here, $[i]^{(k)}$ represents the k th derivative of $[i]$ with respect to q , and $[i]$ is the q -analog of the number i , namely $1 + q + \dots + q^{i-1}$.

And then E_1 becomes the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & [2] & 0 & 0 & 0 & \dots \\ 1 & [3]' & [3] & 0 & 0 & \dots \\ 1 & \frac{[4]''}{2} & [4]' & [4] & 0 & \dots \\ 1 & \frac{[5]'''}{6} & \frac{[5]''}{2} & [5]' & [5] & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let W_1 be the (row) vector $(1, 0, 0, \dots)$ and V_1 be the (column) vector $(1, 1, 1, \dots)$. It is now easy to check that the required relations hold.

Lemma 2.5. *With the definitions of D_1, E_1, V_1, W_1 above, the following relations hold: $D_1E_1 - qE_1D_1 = D_1 + E_1$, $DV_1 = \frac{1}{\beta}V_1$, and $W_1E = \frac{1}{\alpha}W_1$.*

Proof. First note that

$$(D_1E_1)_{i,j} = \beta^{-1}(E_1)_{i+1,j} = \beta^{i-j} \left(\alpha^{-1}q^{j-1}\binom{i}{j-1} + \sum_{r=0}^{j-2}\binom{i-j+r+1}{r}q^r \right),$$

when $j \leq i + 1$, and is equal to 0 otherwise. Then note that

$$q(E_1D_1)_{i,j} = q\beta^{-1}(E_1)_{i,j-1} = \beta^{i-j}q \left(\alpha^{-1}q^{j-2}\binom{i-1}{j-2} + \sum_{r=0}^{j-3}\binom{i-j+r+1}{r}q^r \right).$$

when $1 \leq j - 1 \leq i$, and is equal to 0 otherwise.

Putting these together, we find that $(D_1 E_1 - q E_1 D_1)_{i,j}$ is equal to

$$\begin{aligned}
&= \beta^{i-j} \left(\alpha^{-1} q^{j-1} \binom{i}{j-1} + \sum_{r=0}^{j-2} \binom{i-j+r+1}{r} q^r \right) - \\
&\quad \beta^{i-j} \left(\alpha^{-1} q^{j-1} \binom{i-1}{j-2} + \sum_{r=0}^{j-3} \binom{i-j+r+1}{r} q^{r+1} \right) \\
&= \beta^{i-j} \alpha^{-1} q^{j-1} \binom{i-1}{j-1} + \beta^{i-j} \sum_{r=0}^{j-2} \binom{i-j+r+1}{r} q^r - \beta^{i-j} \sum_{s=1}^{j-2} \binom{i-j+s}{s-1} q^s \\
&= \beta^{i-j} \alpha^{-1} q^{j-1} \binom{i-1}{j-1} + \beta^{i-j} \sum_{r=0}^{j-2} \binom{i-j+r}{r} q^r,
\end{aligned}$$

when $1 < j \leq i$. Also, we have that $(D_1 E_1 - q E_1 D_1)_{i,j} = \beta^{-1}$ when $j = i + 1$, is equal to $\alpha^{-1} \beta^{-1}$ when $j = 1$, and is equal to 0 when $j > i + 1$. These are precisely the matrix entries of $D_1 + E_1$. \square

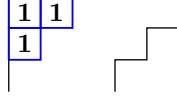
As we will show in Sections 3 and 4 the matrix product $W_1(\prod_{i=1}^n (\tau_i D_1 + (1 - \tau_i) E_1)) V_1$ from Theorem 2.3 has a combinatorial interpretation as a generating function for permutation tableaux.

Example 2.6. Suppose that we are considering the PASEP model with 3 sites, and are interested in computing the probability that in the long time limit, the system reaches the configuration which has a black particle in the second site (but the first and third sites are empty). This configuration is represented by the vector $\tau = (0, 1, 0)$. Using Theorem 2.3, we see that $f_3(0, 1, 0)$ is equal to the matrix product $W_1(E_1 D_1 E_1) V_1$; this product is $\alpha^{-2} + \alpha^{-1} \beta^{-1} + \alpha^{-2} \beta^{-1} q$. And by Remark 2.4, the partition function Z_3 is equal to $W_1(D_1 + E_1)^3 V_1$, which is in this case $\alpha^{-3} + 2\alpha^{-2} + 2\alpha^{-1} + \alpha^{-2} \beta^{-1} + 2\alpha^{-1} \beta^{-1} + 2\beta^{-1} + \alpha^{-1} \beta^{-2} + 2\beta^{-2} + \beta^{-3} + q(\alpha^{-2} + \alpha^{-2} \beta^{-1} + 4\alpha^{-1} \beta^{-1} + \alpha^{-1} \beta^{-2} + \beta^{-2}) + q^2(\alpha^{-2} \beta^{-1} + \alpha^{-1} \beta^{-2})$. Therefore the probability that in the long time limit the system is in the configuration $(0, 1, 0)$ is $\frac{\alpha^{-2} + \alpha^{-1} \beta^{-1} + \alpha^{-2} \beta^{-1} q}{Z_3}$.

3. CONNECTION WITH PERMUTATION TABLEAUX

Recall that a *partition* $\lambda = (\lambda_1, \dots, \lambda_k)$ is a weakly decreasing sequence of non-negative integers. For a partition λ , where $\sum \lambda_i = m$, the *Young diagram* Y_λ of shape λ is a left-justified diagram of m boxes, with λ_i boxes in the i -th row. We define the *expanse* of λ or Y_λ to be the sum of the number of rows and the number of columns. Note that we will allow a row to have length 0—i.e. we allow $\lambda_i = 0$ —and we distinguish two partitions that differ in their number of empty rows.

We will often identify a Young diagram Y_λ with expanse n with the lattice path $p(\lambda)$ of length n which takes unit steps south and west, beginning at the north-east corner of Y_λ and ending at the south-west corner. Note that such a lattice path always begins with a step south. See Figure 2 for the path corresponding to the Young diagram of shape $(2, 1, 0)$.

FIGURE 2. Young diagram and path for $\lambda = (2, 1, 0)$

If $\tau \in \{0, 1\}^{n-1}$, then we associate to it a Young diagram $\lambda(\tau)$ with expanse n as follows. First we define a path $p = (p_1, \dots, p_n) \in \{S, W\}^n$ such that $p_1 = S$, and $p_{i+1} = S$ if and only if $\tau_i = 1$. We then define $\lambda(\tau)$ to be the partition associated to this path p . We denote by ϕ the inverse of the above map: it is a bijection from the set of Young diagrams with expanse n to the set of $n - 1$ -tuples in $\{0, 1\}^{n-1}$.

As in [16], we define a *permutation tableau* \mathcal{T} to be a partition λ together with a filling of the boxes of Y_λ with 0's and 1's such that the following properties hold:

- (1) Each column of the rectangle contains at least one 1.
- (2) There is no 0 which has a 1 above it in the same column *and* a 1 to its left in the same row.

We call such a filling a *valid* filling of Y_λ .

Note that the second requirement above can be rephrased in the following way. Read the columns of a permutation tableau \mathcal{T} from right to left. If in any column we have a 0 which lies beneath some 1, then all entries to the left of 0 (which are in the same row) must also be 0's.

Note that if we forget the requirement (1) above we recover the definition of a J-diagram [12, 17], an object which represents a cell in the totally nonnegative part of the Grassmannian.

We will now define a few statistics on permutation tableaux. We define the *weight* $\text{wt}(\mathcal{T})$ of a permutation tableau \mathcal{T} with k columns to be the total number of 1's in the filling minus k . (We subtract k since there must be at least k 1's in a valid filling of a tableau with k columns.) In other words, we are counting how many extra 1's each column contains beyond the requisite one.

We define $f(\mathcal{T})$ to be the number of 1's in the first row of \mathcal{T} .

We say that an entry in a column of a permutation tableau is *restricted* if that entry is a 0 which lies below some 1. And we say that a row is *unrestricted* if it does not contain a restricted entry. Define $u(\mathcal{T})$ to be the number of unrestricted rows of \mathcal{T} minus 1. (We subtract 1 since the top row of a tableau is always unrestricted.)

Figure 3 gives an example of a permutation tableau \mathcal{T} with weight $19 - 10 = 9$ and expanse 17, such that $u(\mathcal{T}) = 3$ and $f(\mathcal{T}) = 5$.

Now we will be interested in the question of enumerating permutation tableaux according to their shape, weight, unrestricted rows, and first row. That is, we are interested in computing the polynomials $F_\lambda(q) := \sum_{\mathcal{T}} q^{\text{wt}(\mathcal{T})} \alpha^{-f(\mathcal{T})} \beta^{-u(\mathcal{T})}$, where the sum ranges over all permutation tableaux \mathcal{T} of shape λ . Let us also define the polynomials $F^n(q) := \sum_{\mathcal{T}} q^{\text{wt}(\mathcal{T})} \alpha^{-f(\mathcal{T})} \beta^{-u(\mathcal{T})}$, where the sum ranges over all permutation tableaux \mathcal{T} with expanse n .

Our main result is the following.

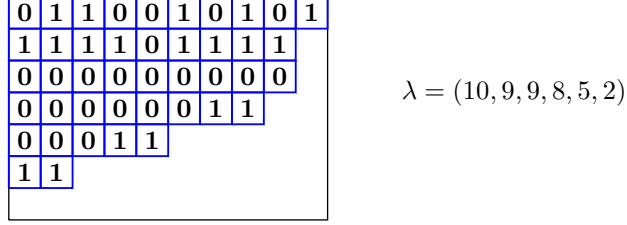


FIGURE 3. A permutation tableau

Theorem 3.1. *Fix a partition λ with expanse $n + 1$. Let $(\tau_1, \dots, \tau_n) \in \{0, 1\}^n$ be $\phi(\lambda)$. Then*

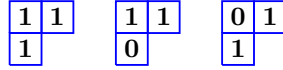
$$F_\lambda(q) = W_1\left(\prod_{i=1}^n (\tau_i D_1 + (1 - \tau_i) E_1)\right) V_1.$$

Moreover, the generating function $F_\lambda^{n+1}(q)$ for all permutation tableaux with expanse $n + 1$ is $W_1(D_1 + E_1)^n V_1$.

An examination of the formula in Theorem 3.1 reveals that $F_\lambda(q)$ is just the sum of the entries in the top row of a certain product of n matrices, each one either D_1 or E_1 , where this product corresponds via ϕ to the shape of the partition λ .

We remark that Theorem 3.1 is particularly nice because it is a *positive* formula. In [16], a recurrence was given for $F_\lambda(q)$ (in the case where $\alpha = \beta = 1$), but not an explicit formula.

Example 3.2. *We now illustrate Theorem 3.1 with an example. Suppose that we want to calculate the weight generating function for permutation tableaux of shape $\lambda = (2, 1)$. Note that $\phi(\lambda) = (0, 1, 0)$. Therefore we need to evaluate the expression $W_1 E_1 D_1 E_1 V_1$. This is equal to $\alpha^{-2} \beta^{-1} q + \alpha^{-2} + \alpha^{-1} \beta^{-1}$. And indeed, as shown in Figure 4, there are three permutation tableaux of shape $(2, 1)$, whose statistics correspond to the three terms above. Compare our results here to Example 2.6.*

FIGURE 4. Permutation tableaux of shape $(2, 1, 0)$

We will defer the proof of Theorem 3.1 to Section 4, in order to first explain various consequences of the result.

Theorem 3.1 together with Theorem 2.3 implies the following.

Corollary 3.3. *Fix $\tau = (\tau_1, \dots, \tau_n) \in \{0, 1\}^n$, and let $\lambda := \lambda(\tau)$. (Note that $\text{expanse}(\lambda) = n + 1$.) The probability of finding the PASEP model in configuration*

(τ_1, \dots, τ_n) in the steady state is

$$\frac{F_\lambda(q)}{Z_n}.$$

Here, $F_\lambda(q)$ is the weight-generating function for permutation tableaux of shape λ . Moreover, the partition function Z_n for the PASEP model is equal to the generating function for all permutation tableaux with expanse $n+1$, enumerated according to weight, unrestricted rows, and the first row.

In the case that $\alpha = \beta = 1$, this corollary together with results from work with Steingrímsson [16] will give us two more combinatorial interpretations of the probabilities in the steady state, in terms of excedences and crossings, and descents and patterns in permutations. We now recall some definitions and results.

If $\pi \in S_n$ is a permutation, we say that π has a *weak excedence* in position i if $\pi(i) \geq i$. Note that there is always a weak excedence in position 1. The *weak excedence set* of π is the set of all indices i where there is a weak excedence. Additionally, we say that π has a *crossing* in positions (i, j) if either $j < i \leq \pi(j) < \pi(i)$ or $\pi(i) < \pi(j) < i < j$. Crossings were first defined by the first author in [5].

Theorem 3.4. [16, Theorem 7] *Let $T(k, n, c)$ be the set of permutation tableaux with k rows, $n - k$ columns, and weight c . Let $M(k, n, c)$ be the set of all permutations $\pi \in S_n$ with k weak excedences and c crossings. Then there is a bijection $\Phi : T(k, n, c) \rightarrow M(k, n, c)$. Moreover, Φ maps a tableau \mathcal{T} whose path $\phi(\mathcal{T})$ has south steps in positions $S \subset \{1, \dots, n\}$ to a permutation with weak excedence set S .*

Given $\tau = (\tau_1, \dots, \tau_n) \in \{0, 1\}^n$, we define $W(\tau)$ to be the subset of $\{1, 2, \dots, n+1\}$ which contains 1 and also contains $i+1$ if and only if $\tau_i = 1$. Define the *excedence weight* $\text{wt}(\pi)$ to be the number of crossings of π , and define $F'_\tau(q) := \sum_{\pi} q^{\text{wt}(\pi)}$, where the sum is over all permutations in S_{n+1} with weak excedence set $W(\tau)$.

By combining Corollary 3.3 and Theorem 3.4, we obtain the following result.

Corollary 3.5. *Fix $\tau = (\tau_1, \dots, \tau_n) \in \{0, 1\}^n$ and suppose that $\alpha = \beta = 1$. The probability of finding the PASEP model in configuration (τ_1, \dots, τ_n) in the steady state is*

$$\frac{F'_\tau(q)}{Z_n}.$$

Here $F'_\tau(q)$ is the generating function which enumerates permutations in S_{n+1} with weak excedence set $W(\tau)$ according to number of crossings.

This corollary can also be obtained in another way. Namely, in [3], the authors provide a lattice path interpretation for the probability of find the PASEP model in a particular configuration. If one then applies a certain bijection on lattice paths [5, Lemma 9] and then the bijection of Foata and Zeilberger [10] from lattice paths to permutations, one arrives at Corollary 3.5 in this manner.

Example 3.6. *Suppose that we are interested in $\tau = (0, 1, 0)$. If we set $\alpha = \beta = 1$ and use the results of Example 2.6, then we find that the probability of finding the PASEP model in configuration τ in the steady state (for $\alpha = \beta = 1$) is $\frac{q+2}{2q^2+8q+14}$. Note that $W(\tau) = (1, 3)$. Now observe that there are precisely three permutations*

in S_4 with weak excedence set $(1, 3)$: $(3, 1, 4, 2)$, and $(4, 1, 3, 2)$, and $(2, 1, 4, 3)$. The first of these permutations has one crossing and the second two have none, so the generating function for permutations of type τ is $q+2$, in agreement with Corollary 3.5.

We now introduce some more definitions concerning permutations. If $\pi \in S_n$ is a permutation, we say that π has a *descent* in position i if $\pi(i) > \pi(i+1)$. The *descent set* of π is the subset of $\{1, 2, \dots, n-1\}$ where π has descents. And we say that π has an occurrence of a *generalized pattern of type 2-31* if there is a pair (i, j) such that $1 \leq i < j < n$ and $\pi(j+1) < \pi(i) < \pi(j)$. These patterns were defined by Babson and Steingrímsson [1].

Given $\tau = (\tau_1, \dots, \tau_n) \in \{0, 1\}^{n-1}$, we define $D(\tau)$ to be the subset of $\{1, 2, \dots, n\}$ which contains i if and only if $\tau_i = 1$. We define the *descent weight* $\text{wt}'(\pi)$ to be the number of crossings of π , and define $F_\tau''(q) := \sum_{\pi} q^{\text{wt}'(\pi)}$, where the sum is over all permutations in S_{n+1} with descent set $D(\tau)$.

We need the following result from [16], which is also proved in [5, Proposition 6].

Theorem 3.7. [16, Theorem 16]. *There is a bijection Ψ on permutations in S_n , which sends a permutation π with descent set $D(\tau)$ to a permutation $\Psi(\pi)$ with weak excedence set $W(\tau)$. Moreover, the excedence weight of $\Psi(\pi)$ is equal to the descent weight of $\Psi(\pi)$, i.e. the number of crossings of $\Psi(\pi)$ is equal to the number of occurrences of the pattern 2-31 in π .*

By applying this result to Corollary 3.5, we get the following.

Corollary 3.8. *Fix $\tau = (\tau_1, \dots, \tau_n) \in \{0, 1\}^n$ and suppose $\alpha = \beta = 1$. The probability of finding the PASEP model in configuration (τ_1, \dots, τ_n) in the steady state is*

$$\frac{F_\tau''(q)}{Z_n}.$$

Here $F_\tau''(q)$ is the generating function which enumerates permutations in S_{n+1} with descent set $D(\tau)$ according to the number of occurrences of the pattern 2-31.

Now we will show how these results imply a result of the first author [5].

First we recall the definition of the polynomials $\hat{E}_{k,n}(q)$ which were introduced by the second author [17]. Define

$$\hat{E}_{k,n}(q) = q^{k-k^2} \sum_{i=0}^{k-1} (-1)^i [k-i]^n q^{ki-k} \left(\binom{n}{i} q^{k-i} + \binom{n}{i-1} \right).$$

It was shown (implicitly) there (and more explicitly in [16]) that $\hat{E}_{k,n}(q)$ enumerates permutation tableaux with k rows and $n-k$ columns according to weight. Additionally, it was shown in [17] that at $q = -1, 0, 1$, $\hat{E}_{k,n}(q)$ specializes to binomial coefficients, Naryana numbers, and Eulerian numbers.

In [5], the following connection was made between these polynomials and the PASEP model.

Theorem 3.9. [5] *Let $\alpha = \beta = 1$. Then in the steady state, the probability that the PASEP model with n sites is in a configuration with precisely k particles is:*

$$\frac{\hat{E}_{k+1,n+1}(q)}{Z_n}$$

Since the polynomials $\hat{E}_{k,n}(q)$ enumerate permutation tableaux with k rows according to weight, Corollary 3.3 implies Theorem 3.9.

4. PROOF OF THEOREM 3.1

Recall the following requirement for permutation tableaux. When we read the columns of a permutation tableau \mathcal{T} from right to left, if in any column we have a 0 which lies beneath some 1, then all entries to the left of 0 (which are in the same row) must also be 0's.

Also recall that an entry in a column of a permutation tableau is *restricted* if that entry is a 0 which lies below some 1. And a row is *unrestricted* if it does not contain a restricted entry.

Proof. We will prove Theorem 3.1 inductively, by finding a precise combinatorial interpretation for each of the entries in the top row of a matrix product such as

$$\prod_{i=1}^n (\tau_i D_1 + (1 - \tau_i) E_1).$$

Let $F_\lambda^i(q) := \sum_{\mathcal{T}} q^{\text{wt}(\mathcal{T})} \alpha^{-f(\mathcal{T})} \beta^{-u(\mathcal{T})} = \sum_{\mathcal{T}} q^{\text{wt}(\mathcal{T})} \alpha^{-f(\mathcal{T})} \beta^{-i+1}$, where the sum ranges over all permutation tableaux \mathcal{T} of shape λ which have precisely i unrestricted rows. Clearly $F_\lambda(q) = \sum_{i \geq 1} F_\lambda^i(q)$. Note that a permutation tableau will always have at least one unrestricted row (the top one).

Now let M_λ be the matrix $\prod_{i=1}^n (\tau_i D_1 + (1 - \tau_i) E_1)$, where $(\tau_1, \dots, \tau_n) = \phi(\lambda)$. We claim that the entry $M_\lambda[1, i]$ in position $(1, i)$ of M_λ is $F_\lambda^i(q)$.

First note that this claim holds when M_λ is equal to D_1 or E_1 . M_λ is equal to D_1 when λ is the partition $(0, 0)$. In this case there is a single permutation tableaux with two rows and no columns, which has weight 0, and two unrestricted rows (hence $f(\mathcal{T}) = 0$ and $u(\mathcal{T}) = 1$). This corresponds to the fact that the top row of D_1 is $(0, \beta^{-1}, 0, \dots)$. Similarly, M_λ is equal to E_1 when λ is the partition (1) . In this case there is a single permutation tableaux with one box which is filled with a 1 (hence $f(\mathcal{T}) = 1$). This tableaux has weight 0 and one unrestricted row (hence $u(\mathcal{T}) = 0$), corresponding to the fact that the top row of E_1 is $(\alpha^{-1}, 0, 0, \dots)$.

Using induction, assume that our claim is true for λ with expanse less than or equal to n . In other words, we can interpret the i th entry of the top row of M_λ as a generating function enumerating permutation tableaux of shape λ with i unrestricted rows, according to weight. Let us now consider how these generating functions will change if we instead consider permutation tableaux of shape λ' , where λ' is a new shape obtained from λ by either adding a new row of length 0, or a new column whose length is the number of rows (including rows of length 0) of λ . The corresponding operation on paths is the following: we take the path corresponding

to λ and add an additional step from its south-west corner which is either south or west.

It is easy to see how adding a step south to the partition path — i.e. adding an empty row to λ — will affect the generating functions. Any permutation tableau \mathcal{T} of the new shape λ' will be a permutation tableau of shape λ with one additional unrestricted row (the last row). Therefore if λ' is equal to λ union a row of length 0, we will have $F_{\lambda'}^1(q) = 0$, and $F_{\lambda'}^{i+1}(q) = \beta^{-1} F_{\lambda}^i(q)$ for $i \geq 1$. This corresponds to the fact that in the matrix product $M_{\lambda} D_1$, the new top row will be $(0, \beta^{-1} M_{\lambda}[1, 1], \beta^{-1} M_{\lambda}[1, 2], \dots)$.

It now remains to see how adding a step west to the partition path — i.e. adding an extra column of length r to the left-hand-side of $\lambda = (\lambda_1, \dots, \lambda_r)$ — will affect the generating functions. In this case our new partition λ' is equal to $(\lambda_1 + 1, \dots, \lambda_r + 1)$.

Let $h_{a,b}(q)$ denote the polynomial $\beta^{b-a} \left(\alpha^{-1} q^{a-1} \binom{b-1}{a-1} + \sum_{j=0}^{a-2} q^j \binom{b-a+j}{j} \right)$ for $a \leq b$. We claim that $F_{\lambda'}^a(q) = \sum_{b \geq a} h_{a,b}(q) F_{\lambda}^b(q)$.

To prove this, it is enough to show the following. Fix a permutation tableau \mathcal{T} of shape λ which has precisely b unrestricted rows. Consider all ways of adding an additional (maximal) column C to the left of \mathcal{T} , in order to build a new permutation tableau \mathcal{T}' of shape λ' which has precisely a unrestricted rows. Then the generating function for these new tableaux according to weight is precisely $h_{a,b}(q) q^{\text{wt}(\mathcal{T})}$.

To prove this last statement, consider the process of adding the column C to \mathcal{T} . Since \mathcal{T} has b unrestricted rows, there are only b entries in C in which we can choose to put either a 1 or 0; all other entries are forced to be 0. Let us number those unrestricted entries from top to bottom by c_1 to c_b . Since we want our new column C to add an additional $b - a$ restricted positions, in our filling of the entries c_1, \dots, c_b , we must have precisely $b - a$ 0's below a 1. If the top-most 1 in our filling of c_1, \dots, c_b is in position c_i , then the $i - 1$ entries above it are all 0's, and we have precisely $\binom{b-i}{b-a}$ ways to choose which entries to make 0 below c_i . The other $(b - i) - (b - a) = a - i$ entries must be 1's. This particular choice of column will therefore contribute the extra weight q^{a-i} to the weight of \mathcal{T} — if $i \neq 1$ — and will contribute the weight $\alpha^{-1} q^{a-1}$ — if $i = 1$. If we sum over all possible columns C which we may add to \mathcal{T} , we get the following: $\alpha^{-1} q^{a-1} \binom{b-1}{b-a} + \sum_{i=2}^a q^{a-i} \binom{b-i}{b-a}$ which is equal to $\alpha^{-1} q^{a-1} \binom{b-1}{a-1} + \sum_{j=0}^{a-2} q^j \binom{b-a+j}{j}$. This completes the proof that $F_{\lambda'}^a(q) = \sum_{b \geq a} h_{a,b}(q) F_{\lambda}^b(q)$. And now note that this corresponds to the fact that in the matrix product $M_{\lambda} E_1$, the entry in the first row and a th column will be $\sum_{b \geq a} h_{a,b}(q) M_{\lambda}[1, b]$.

This now completes our proof that for any partition λ , the entry $M_{\lambda}[1, i]$ in position $(1, i)$ of M_{λ} is $F_{\lambda}^i(q)$. And since $F_{\lambda}(q) = \sum_{i \geq 1} F_{\lambda}^i(q)$, the theorem follows. \square

5. APPLICATIONS OF PERMUTATION TABLEAUX

In this section we will give some applications of the connection of permutation tableaux to the PASEP. The first application is a partial order on states of the

PASEP and some monotonicity results (with respect to that partial order) of probabilities of observing these states. Note that some of the results below hold for general α, β ($0 < \alpha \leq 1$ and $0 < \beta \leq 1$), and for others we need to assume that $\alpha = \beta = 1$.

Definition 5.1. Let $\tau, \tau' \in \{0, 1\}^n$ be two states of the PASEP which contain exactly k particles. We define the partial order \prec by $\tau \prec \tau'$ if and only if $\lambda(\tau) \subset \lambda(\tau')$.

Figure 5 illustrates this partial order when $n = 5$ and $k = 2$, and it shows the (unnormalized) probabilities $f_n(\tau)$ that each of these states occurs, for $\alpha = \beta = 1$.

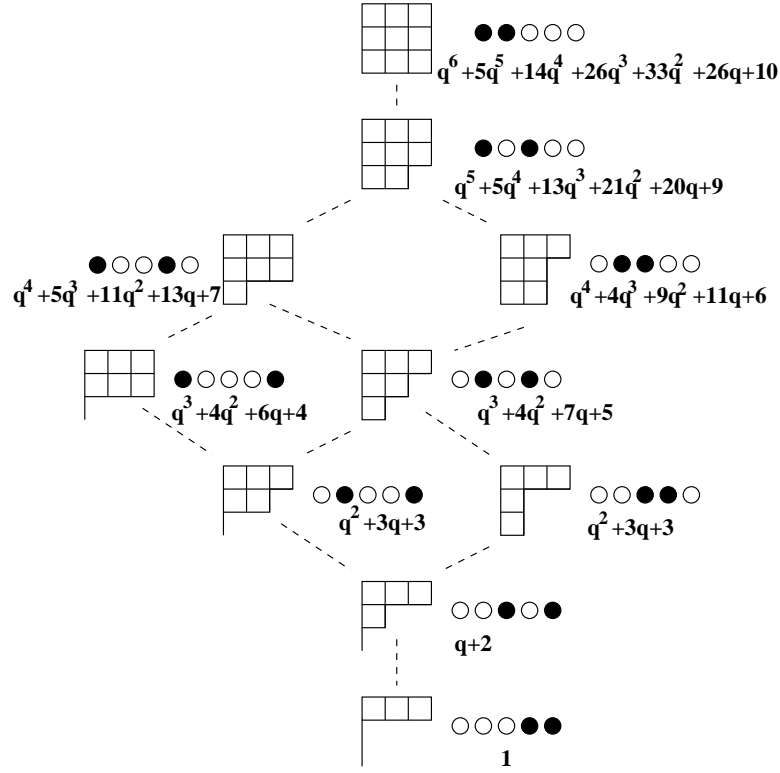


FIGURE 5. A partial order for states of the PASEP

We now give two simple inequalities relating $f_n(\tau)$ to $f_n(\tau')$ when $\tau \prec \tau'$.

Proposition 5.2. Let α and β be general. Suppose that $\tau \prec \tau'$ and let $d := |\lambda(\tau')| - |\lambda(\tau)|$. That is, d is the difference between the cardinalities of the Young diagrams $\lambda(\tau')$ and $\lambda(\tau)$. Then $f_n(\tau') - q^d f_n(\tau)$ is a non-negative polynomial in q , α^{-1} , and β^{-1} .

Proof. Observe that any permutation tableau of shape τ can be naturally extended to a permutation tableau of shape τ' by filling in all boxes of $\lambda(\tau') \setminus \lambda(\tau)$ with 1's. This new permutation tableau has weight q^d times the weight of the old one. \square

Proposition 5.3. *Let $\alpha = \beta = 1$. Suppose that $\tau \prec \tau'$. Then $f_n(\tau') - f_n(\tau)$ is a non-negative polynomial in q . In other words, as one moves up the partial order \prec , the coefficients of $f_n(\tau)$ monotonically increase.*

Proposition 5.3 is false for general α and β . For example, $f_4(1, 1, 0, 0) - f_4(1, 0, 1, 0)$ is a Laurent polynomial in q , α , and β with some negative coefficients.

We will defer the proof of Proposition 5.3 to the next section because it is most easily proved using Motzkin paths.

We thank the referee for pointing out that the corollary below follows from our previous results.

Corollary 5.4. *Let $\alpha = \beta = 1$. Suppose that $\tau \prec \tau'$. Let d be any integer such that $0 \leq d \leq |\lambda(\tau')| - |\lambda(\tau)|$. Then $f_n(\tau') - q^d f_n(\tau)$ is a non-negative polynomial in q .*

Proof. This follows from Propositions 5.2 and 5.3. Simply choose a saturated chain of partitions $\tau = \tau_0 \prec \tau_1 \prec \dots \tau_m = \tau'$ such that $|\lambda(\tau_{k+1})| - |\lambda(\tau_k)| = 1$ and apply either Proposition 5.2 or Proposition 5.3 at each step. \square

The above results make sense intuitively, since in a model with particles entering from the left and leaving to the right, it is more likely that a given particle will be further to the left than to the right. Also recall that the probability of hopping left ($\frac{q}{n+1}$ for some $q \leq 1$) is at most the probability of hopping right ($\frac{1}{n+1}$).

Another application of permutation tableaux is that these objects allow one to read off the main recurrences for the PASEP ([3, Theorem 1]) quite easily. The rest of the results in this section hold for general α and β .

Fix a partition $\lambda = (\lambda_1, \dots, \lambda_k)$, and let $\tau = (\tau_1, \dots, \tau_n)$ be the unique vector in $\{0, 1\}^n$ such that $\lambda = \lambda(\tau)$. Choose any *corner* of λ , i.e. the last box of a row λ_i in λ such that $\lambda_{i+1} < \lambda_i$. (Equivalently, a pair $(j, j+1)$ of entries in τ such that $\tau_j = 1$ and $\tau_{j+1} = 0$.) As Figure 6 illustrates, there is a simple recurrence for $F_\lambda(q)$.

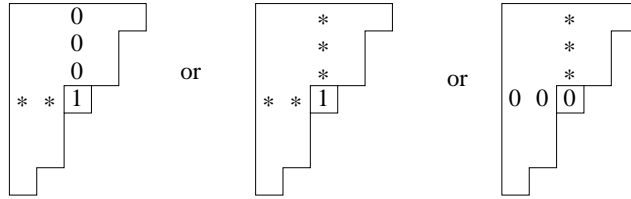


FIGURE 6. Recurrence for $F_\lambda(q)$

Explicitly, any valid filling of λ is obtained in one of the following ways:

- inserting a column whose bottom entry is 1 and whose other entries are 0 after the $(\lambda_i - 1)$ st column of a valid filling of $(\lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_k - 1)$;
- adding a 1 to the end of the i th row of a valid filling of the shape $(\lambda_1, \lambda_2, \dots, \lambda_i - 1, \dots, \lambda_k)$;

- inserting an all-zero row of length λ_i as the i th row in a valid filling of $(\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_k)$.

Note that none of the operations above change the number of 1's in the first row or the number of unrestricted rows. Therefore, for $1 < i \leq k$ and $\lambda_i > \lambda_{i+1}$, we have that

$$F_\lambda(q) = F_{(\lambda_1-1, \dots, \lambda_i-1, \lambda_{i+1}, \dots, \lambda_k)} + F_{(\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_k)} + qF_{(\lambda_1, \dots, \lambda_{i-1}, \lambda_i-1, \lambda_{i+1}, \dots, \lambda_k)}.$$

By Corollary 3.3, this recurrence translates into a recurrence for the PASEP model. Recall our notation $\frac{f_n(\tau_1, \dots, \tau_n)}{Z_n}$ for the probability of finding the system in configuration (τ_1, \dots, τ_n) in the steady state.

Corollary 5.5. [3, Theorem 1] *Let α and β be general. Then*

$$\begin{aligned} f_n(\tau_1, \tau_2, \dots, \tau_{j-1}, 1, 0, \tau_{j+2}, \dots, \tau_n) &= f_{n-1}(\tau_1, \tau_2, \dots, \tau_{j-1}, 1, \tau_{j+2}, \dots, \tau_n) + \\ &\quad qf_n(\tau_1, \tau_2, \dots, \tau_{j-1}, 0, 1, \tau_{j+2}, \dots, \tau_n) + \\ &\quad f_{n-1}(\tau_1, \dots, \tau_{j-1}, 0, \tau_{j+2}, \dots, \tau_n). \end{aligned}$$

It is also easy to prove the following, by consideration of permutation tableaux.

Corollary 5.6. [3, Theorem 1] *Let α and β be general. We have that $f_n(\tau_1, \dots, \tau_{n-1}, 1) = \frac{1}{\beta}f_n(\tau_1, \dots, \tau_{n-1})$ and also $f_n(0, \tau_1, \tau_2, \dots, \tau_n) = \frac{1}{\alpha}f_n(\tau_1, \tau_2, \dots, \tau_n)$.*

Proof. This also follows from Corollary 3.3. The first equality holds because the generating function for permutation tableaux of shape $(\lambda_1, \lambda_2, \dots, \lambda_{n-1}, 0)$ is clearly equal to the generating function for permutation tableaux of shape $(\lambda_1, \lambda_2, \dots, \lambda_{n-1})$ except that there is one additional unrestricted row. The second equality holds because the generating function for permutation tableaux of shape $(\lambda_1 + 1, \lambda_2, \dots, \lambda_n)$ is equal to the generating function for permutation tableaux of shape $(\lambda_1, \lambda_2, \dots, \lambda_n)$ except that there is one additional entry in the first row which is a 1. (Recall that each column of a permutation tableaux is required to contain exactly one 1 and that the weight of a column is equal to the number of 1's it contains *beyond* that requisite one.) \square

6. CONNECTION WITH MOTZKIN PATHS

We now give another solution to the matrix ansatz, which is essentially the one described in Derrida et al [7], and which has an interpretation in terms of Motzkin paths.

Note that in this section we restrict to the case $\alpha = \beta = 1$. These results can be extended to the more general case, but we restrict ourselves to this case since our main purpose here is to obtain the result needed for the monotonicity result of the previous section.

Let D_0 be the (infinite) upper triangular matrix (d_{ij}) such that $d_{i,i} = [i]$ where $[i]$ is the q -analogue of i , $d_{i,i+1} = [i+1]$, and $d_{i,j} = 0$ for $j \neq i, i+1$.

That is, D_0 is the matrix

$$\begin{pmatrix} [1] & [2] & 0 & 0 & \dots \\ 0 & [2] & [3] & 0 & \dots \\ 0 & 0 & [3] & [4] & \dots \\ 0 & 0 & 0 & [4] & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let E_0 be the (infinite) lower triangular matrix $(e_{i,j})$ such that $e_{i,i} = [i]$, $e_{i+1,i} = [i]$, and $e_{i,j} = 0$ for $j \neq i, i-1$.

That is, E_0 is the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & [2] & 0 & 0 & \dots \\ 0 & [2] & [3] & 0 & \dots \\ 0 & 0 & [3] & [4] & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let W_0 be the (row) vector $(1, 0, 0, \dots)$ and V_0 be the (column) vector $(1, 0, 0, \dots)$. It is now easy to check that $D_0 E_0 - q E_0 D_0 = D_0 + E_0$, $D V_0 = V_0$, and $W_0 E = W_0$.

We will now give an interpretation of the steady states of the PASEP in terms of bicolored Motzkin paths. This result is very similar to a result obtained in Brak et al [3], via a combinatorial derivation. Note also that in [2], Brak and Essam considered the case $q = 0$ of the PASEP and gave multiple interpretations for the steady states in terms of various weighted lattice paths.

We define a *bicolored Motzkin path* of length n to be a sequence of steps in the plane $c = (c_1, \dots, c_n)$ such that $c_i \in \{N, S, E, \bar{E}\}$ for $1 \leq i \leq n$, which starts and ends at height 0, and always stays at or above height 0. That is, if we define the height at the i th step to be $h_i = \{j < i \mid c_j = N\} - \{j < i \mid c_j = S\}$, then $h_i \geq 0$ for $1 \leq i \leq n$, and furthermore, $h_{n+1} = 0$.

We now assign a weight $\text{wt}(p)$ to a bicolored Motzkin path p as follows: step c_i is assigned weight $[h+1]$ if c_i ends at height h , and the weight of c is defined to be the product of the weights of all the c_i 's.

Fix a sequence $\tau = (\tau_1, \dots, \tau_n) \in \{0, 1\}^n$. We say that a bicolored Motzkin path c has *type* τ if $c_i = N$ or E whenever $\tau_i = 1$ and $c_i = S$ or \bar{E} whenever $\tau_i = 0$.

We now define $F_\tau'''(q)$ to be the generating function for all bicolored Motzkin paths of type τ : that is, $F_\tau'''(q) := \sum_p q^{\text{wt}(p)}$ where the sum is over all bicolored Motzkin paths of type τ .

The main result of this section is the following.

Proposition 6.1. *Choose a sequence $\tau = (\tau_1, \dots, \tau_n) \in \{0, 1\}^n$. Then*

$$F_\tau'''(q) = W_0 \left(\prod_{i=1}^n (\tau_i D_0 + (1 - \tau_i) E_0) \right) V_0$$

Let us make the convention in this section that rows and columns of our matrices are indexed by non-negative integers (including 0). Then Proposition 6.1 says that

the generating function $F_\tau'''(q)$ is equal to the entry in the 0th row and 0th column of the matrix product $\prod_{i=1}^n (\tau_i D_0 + (1 - \tau_i) E_0)$.

Proof. Using the definition of matrix multiplication, we expand $\prod_{i=1}^n (\tau_i D_0 + (1 - \tau_i) E_0)$ as the sum of terms of the form $A_1[1, i_1] A_2[i_1, i_2] \dots A_n[i_{n-1}, 1]$ where the A_j 's are matrices D_0 or E_0 , depending on the sequence τ . It is obvious that these terms correspond to the Motzkin paths of type τ . \square

As before, we can use Proposition 6.1 together with Theorem 2.3 to deduce the following.

Corollary 6.2. *Fix $\tau = (\tau_1, \dots, \tau_n) \in \{0, 1\}^n$. The probability of finding the PASEP model in configuration (τ_1, \dots, τ_n) in the steady state is*

$$\frac{F_\tau'''(q)}{Z_n}.$$

Here $F_\tau'''(q)$ is the generating function for bicolored Motzkin paths of type τ .

We now use this result to prove Proposition 5.3 from the previous section:

Proof. It is sufficient to consider two configurations τ and τ' whose corresponding diagrams $\lambda(\tau)$ and $\lambda(\tau')$ differ by one box. That is, there exists some i such that:

- $\tau_i = 0, \tau_{i+1} = 1$
- $\tau'_i = 1, \tau'_{i+1} = 0$
- $\tau'_j = \tau_j$ for $j \neq i, i+1$.

We now observe that any bicolored Motzkin path p of type τ (with weight w) can be mapped to a bicolored Motzkin path p' of type τ' whose weight w' is coefficient-wise greater than w .

If $p = (c_1, \dots, c_n)$ then we let $p' := (c_1, \dots, c_{i-1}, c_{i+1}, c_i, c_{i+2}, \dots, c_n)$. That is, p' is the path obtained from p by switching the i th and $i+1$ st steps. It is clear that whether we switch SN steps for NS , or SE for ES , or $\bar{E}N$ for $N\bar{E}$, or $\bar{E}\bar{E}$ for $E\bar{E}$, the resulting path is a valid Motzkin path. The weight w' will differ from w by (respectively): replacing a factor $[h-1][h]$ with $[h+1][h]$; replacing a factor $[h-1]^2$ with $[h][h-1]$; replacing a factor $[h][h+1]$ with $[h+1]^2$; nothing. Using the fact that $[n+1] - [n]$ is nonnegative, all of the differences $[h+1][h] - [h-1][h]$, $[h][h-1] - [h-1]^2$, $[h+1]^2 - [h][h+1]$ are nonnegative; therefore in all cases, $w' - w$ will be a nonnegative polynomial. By Corollary 6.2, we are done. \square

Note that one can use the same argument to give another proof of Proposition 5.2 in the case that $\alpha = \beta = 1$. We repeat the same argument but now w' differs from w by: replacing $q[h-1][h]$ with $[h+1][h]$; replacing $q[h-1]^2$ with $[h][h-1]$; and replacing $q[h][h+1]$ with $[h+1]^2$. Then all differences of such polynomials will be nonnegative, since $[n+1] - q[n]$ is nonnegative.

Remark 6.3. *We now summarize our combinatorial interpretations for the steady state probabilities of the PASEP model; these are given by our solutions (D_j, E_j)*

to the matrix ansatz (where $\alpha = \beta = 1$), for $j = 0, 1$. Note that since

$$W_j(\prod_{i=1}^n (\tau_i D_j + (1 - \tau_i) E_j)) V_j$$

describes the steady state probability of the PASEP model for both $j = 0, 1$, these two formulas must be equal. In particular, both of these are formulas for all of the following:

- $F_{\lambda(\tau)}(q)$, the weight-generating function for permutation tableaux of shape $\lambda(\tau)$.
- $F'_\tau(q)$, the generating function for permutations of exceedance type τ , enumerated according to crossings.
- $F''_\tau(q)$, the generating function for permutations of descent type τ , enumerated according to occurrences of 2 – 31.
- $F'''_\tau(q)$, the generating function for weighted bicolored Motzkin paths.

Moreover, the product $W_j(D_j + E_j)^n V_j$ is the weight-generating function for all of the following:

- permutation tableaux with expanse $n + 1$.
- permutations in S_{n+1} , enumerated according to crossings.
- permutations in S_{n+1} , enumerated according to occurrences of 2 – 31.
- weighted bicolored Motzkin paths of length n .

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